JSJ SPLITTINGS FOR ALL ARTIN GROUPS

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ABSTRACT. We prove that an Artin group splits over infinite cyclic subgroups if and only if its defining graph has a separating vertex, and explicitly construct a JSJ decomposition over infinite cyclic subgroups for all Artin groups. This is then used to show that, if two Artin groups are isomorphic, then they have the same set of big chunks, that is, maximal subgraphs without separating vertices. We also deduce acylindrical hyperbolicity for the automorphism groups of many Artin groups, partially answering a question of Genevois in the case of Artin groups.

> We're not splitting atoms here;
> we're trying to entertain people.
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> Boomer Esiason

INTRODUCTION

Artin groups are a rich class of groups generalising braid groups, with strong connections to Coxeter groups. They are defined via the following presentation: given a finite simplicial graph Γ with vertices $V(\Gamma)$, edges $E(\Gamma)$, and for each edge $\{a, b\} \in E(\Gamma)$ a label $m_{ab} \in \mathbb{Z}_{\geqslant 2}$, the associated Artin group A_Γ is presented by

$$\langle V(\Gamma) \mid \mathrm{prod}(a, b, m_{ab}) = \mathrm{prod}(b, a, m_{ab}) \, \forall \{a, b\} \in E(\Gamma) \rangle,$$

where $\mathrm{prod}(u, v, n)$ denotes the prefix of length n of the infinite alternating word $uvuvuv\ldots$. Despite the explicit presentation, very few properties are known to hold for all Artin groups: it is not even known if they are all torsion-free, or satisfy the famous $K(\pi, 1)$-conjecture [Par14].

In this paper we investigate when an Artin group splits over $\mathbb{Z}$, meaning that it admits a graph of group decomposition with infinite cyclic edge groups. Such splittings assumed a central role in the solution of the isomorphism problem for hyperbolic groups (see [Sel95, Bow98, RS97, DG11] among many others); in particular, "maximal" splittings called *JSJ decompositions* were the key tool for understanding automorphisms of hyperbolic groups. In a similar fashion, we also obtain results related to the isomorphism problem for Artin groups, and establish some geometric properties of automorphism groups of Artin groups (see Theorem B and Corollary E, respectively).

Cyclic splittings of Artin groups. Clay [Cla14] proved that a right-angled Artin group (RAAG) on a connected graph Γ splits over $\mathbb{Z}$ if and only if either Γ contains a separating vertex, or Γ has two vertices (since $\mathbb{Z}^2$ splits over $\mathbb{Z}$). We generalise this to all Artin groups:

Theorem A (see Theorem 2.1). *Let Γ be a finite, connected, labelled simplicial graph. The Artin group A_Γ splits over $\mathbb{Z}$ if and only if either Γ has a separating vertex, or Γ has two vertices.*

In other words, if an Artin group on at least three generators splits over $\mathbb{Z}$, then it must have a *visual* splitting, that is, a splitting coming from a decomposition of the graph into two full proper subgraphs overlapping over a vertex. We also have to include the case where Γ consists of a single edge; indeed, such an Artin group (called a dihedral Artin group) splits over $\mathbb{Z}$, as it is a central extension of an infinite cyclic group by a virtually free group (see Example 3.3).

Our proof is similar in spirit to Clay's. However, we had to implement some upgrades, to circumvent those steps which used properties of RAAGs that are not true for all Artin groups.

Isomorphism invariance of big chunks. One of the most widely open questions for Artin groups is the isomorphism problem, which asks for an algorithm which takes two labelled graphs Γ and Γ' as input, and determines if they yield isomorphic Artin groups. This question, which specialises Dehn's Isomorphism Problem to the family of Artin groups, is solved when one restricts to certain subclasses, including RAAGs [Dro87], spherical-type [Par04], and large-type [Vas22], but there is no unified approach. However, isomorphic Artin groups must admit the same splittings over $\mathbb{Z}$, as this is an algebraic property. In this spirit, if $A_\Gamma \cong A_{\Gamma'}$, then one can hope to use Theorem A to recover some decompositions of Γ and Γ' into maximal connected subgraphs without separating vertices, which we call *big chunks*. We do just that in Theorem 5.6, of which we give a streamlined version here. For a labelled graph Γ, let $\mathcal{BC}(\Gamma)$ be the collection of its big chunks.

Theorem B (see Theorem 5.6). *Let Γ and Γ' be finite, connected, labelled simplicial graphs, and let $\varphi \colon A_\Gamma \to A_{\Gamma'}$ be an isomorphism. Then there exists a bijection $\varphi_\# \colon \mathcal{BC}(\Gamma) \to \mathcal{BC}(\Gamma')$, such that for every $\Lambda \in \mathcal{BC}(\Gamma)$ the following hold:*

(1) $A_\Lambda \cong A_{\varphi_\#(\Lambda)}$.
(2) $A_{\varphi_\#(\Lambda)}$ is a conjugate of $\varphi(A_\Lambda)$, possibly unless Λ is a leaf with label 2.
(3) If Λ is an even leaf, then so is $\varphi_\#(\Lambda)$, with the same label.

In other words, isomorphic Artin groups must have isomorphic big chunks, and, among these, the same number of even leaves of every given label. Theorem B is already new for two-dimensional Artin groups, and we believe that our findings, together with [Vas22, Theorem E], could be a key building block for a future solution of the isomorphism problem in this class.

JSJ decompositions, trees of cylinders, and deformation spaces. The key tool in the proof of Theorem B is the existence of a *JSJ decomposition* over cyclic subgroups, in the sense of [GL17]. We refer to Definition 3.2 for the details; for our purposes, it is enough to recall that, given a graph of group decomposition $\mathcal{G}$ of a group G, with cyclic edge groups, if $\mathcal{G}$-vertex groups are elliptic in every G-action on a tree with cyclic edge stabilisers, then $\mathcal{G}$ is a JSJ decomposition. In other words, a JSJ decomposition should be thought of as a "maximal" splitting of G, which means that vertex groups cannot be further split "in a canonical way". Our methods allow us to build a JSJ decomposition over cyclic subgroups of an Artin group A_Γ, whose vertex groups roughly correspond to the big chunks of Γ:

Theorem C (see Theorem 3.9). *Let Γ be a finite, connected, labelled graph on at least three vertices. There is an A_Γ-tree $J(\Gamma)$ which is a JSJ-decomposition for A_Γ over virtually cyclic subgroups.*

The construction of $J(\Gamma)$ is explicit, but we postpone it to Definition 3.7 and only provide an example here (see Figure 1). Crucially, the proof of Theorem 3.9 uses that big chunks on at least three vertices do not split further over $\mathbb{Z}$, as a consequence of Theorem A.
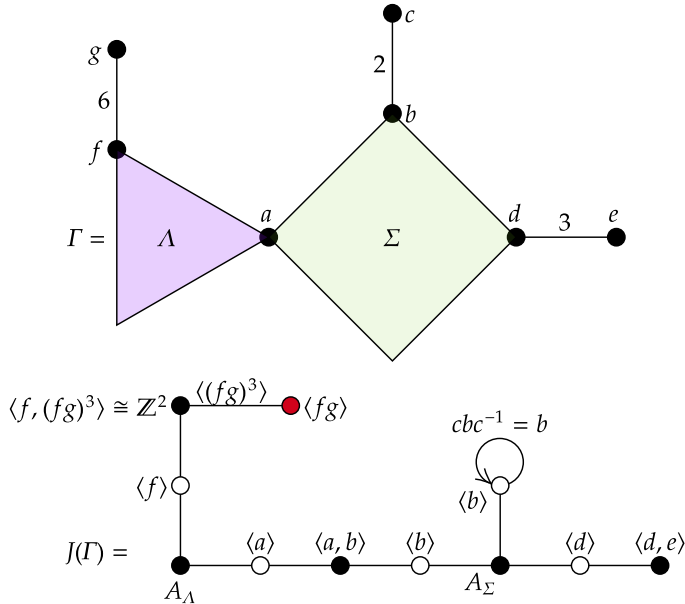

FIGURE 1. The graph Γ and JSJ decomposition $J(\Gamma)$ from Example D.

Example D. Consider a connected graph Γ as in Figure 1, whose big chunks are Σ, Λ, and the three leaves, with labels 2, 3, and 6, respectively. The associated JSJ decomposition $J(\Gamma)$ of A_Γ is also depicted in the Figure. White vertices correspond to separating vertices of Γ, while black vertices roughly correspond to big chunks. The only exception are even leaves: a leaf with label 2 (here, spanned by b and v) further splits as a HNN extension, while an even leaf with higher label (here, generated by f and g) splits as an amalgamation $\mathbb{Z}^2 *_\mathbb{Z} \mathbb{Z}$ over its centre (here, generated by $(fg)^3$, as the label is 6). Whenever an edge of $J(\Gamma)$ is not labelled in this picture, the edge group is the white vertex group, whose embedding in the black vertex group corresponds to the graph inclusion of a separating vertex inside a big chunk.

Our JSJ decomposition is not unique, as different Artin generating sets of the same Artin group may yield non-isomorphic JSJ decompositions. There is a procedure, due to Guirardel and Levitt, to obtain a canonical tree, called the *tree of cylinders*, from the JSJ tree. This tree is left invariant by the automorphism group, and the action turns out to be acylindrical, allowing us to affirmatively answer [Gen19, Question 1.1] for any Artin group whose defining graph contains a separating vertex.

Corollary E (see Corollary 4.6). *Let A_Γ be a torsion-free Artin group, such that Γ is connected and has a separating vertex which does not centralise the whole group. Then $\mathrm{Aut}(A_\Gamma)$ is acylindrically hyperbolic.*

We remark that the assumption that A_Γ is torsion-free is conjecturally trivial, since all Artin groups are expected to be torsion-free. By work of Fournier-Facio and Wade, Corollary E implies that an Artin group A_Γ as in the statement has an infinite-dimensional space of $\mathrm{Aut}(A_\Gamma)$-invariant *quasimorphism* (see [FFW23, Theorem E]). In turn, this has applications to construct unbounded $\mathrm{Aut}(A_\Gamma)$-invariant word norms on A_Γ (see [FFW23, Corollary C]).

The cost of the tree of cylinders is that the edge groups are no longer cyclic. If one wishes to consider cyclic splittings, the canonical object is the *JSJ deformation space*. Deformation spaces were first introduced by Forester [For01], motivated by outer space for free groups [CV86]. Any JSJ decomposition of a group G gives rise to the JSJ space [GL17], a contractible space with an action of the (outer) automorphism group of G. As such, the JSJ deformation space turns out to be a powerful tool to study (outer) automorphism groups; indeed, in [JMS25], we combine the results of this paper with the theory of deformation spaces to tackle a version of the isomorphism problem for new classes of Artin groups. More precisely, we are able to reduce the so-called Strong Twist Conjecture for an Artin group to its big chunks.

The first author recently defined another deformation space for certain large-type Artin groups [Jon24]. We note that, in general, their space does not coincide with the JSJ space of this article.

Future directions. In [GH17], Groves and Hull generalised Clay's results and proved that a RAAG on a connected graph Γ splits over an Abelian subgroup if and only if Γ contains a separating simplex, or is itself complete, to account for $\mathbb{Z}^n$. It is possible that similar techniques may lead to an analogous characterisation of when an Artin group splits over a *spherical* Artin subgroup. Notably, the analogue of Theorem B would yield the isomorphism invariance of *chunk* parabolics, where a chunk is a maximal subgraph without separating cliques of spherical type. This would be relevant since the first author proved that, if the chunks of a large-type Artin group A_Γ are isomorphism invariants and have finite outer automorphism group, then $\mathrm{Out}(A_\Gamma)$ is of type VF, hence finitely presented [Jon24, Theorem 1.1].

The first step in the above direction would be to understand the following:

Question F. Let Γ be a finite, connected, labelled graph, and let DA_n be a dihedral Artin group with label n. What are necessary and sufficient conditions for A_Γ to split over DA_n?

The matter is complicated by the existence of "exotic" maximal dihedral subgroups which are not parabolic (see e.g. [Vas22, Theorem D]). Some of these questions will also be considered in independent work of Sliazkaite.

Organisation of the paper. In Section 1 we recall some properties of simplicial actions on trees. Section 2 is devoted to the characterisation of when an Artin group splits over $\mathbb{Z}$, Theorem A.

In Section 3 we produce a JSJ decomposition, thus proving Theorem C (see Definition 3.7 for the construction of the splitting, and Theorem 3.9 for the proof that it yields a JSJ decomposition). Then in Section 4 we combine the JSJ with

the machinery of [GL11] to prove Corollary E on acylindrical hyperbolicity of automorphism groups of certain Artin groups (see Corollary 4.6).

Finally, the JSJ plays a central role in Section 5, where we prove that big chunks are isomorphism invariant, Theorem B (see Theorem 5.6).

1. Background

We first recall some properties of simplicial actions on trees, referring to [Ser03] for further generalities. We shall work in the following setting:

Notation 1.1. By *tree* we mean a simply connected simplicial graph, equipped with the metric where each edge has length one. Given a group G, a *G-tree* (T, Ω) is a tree T endowed with an action $\Omega \colon G \to \mathrm{Aut}(T)$ by simplicial isometries, without edge inversions, and minimal (i.e. no proper sub-tree is invariant under the action). We often suppress the reference to the action Ω when it is not relevant or it is clear from the context.

Throughout we will write $\mathrm{Stab}_\Omega(S)$ for the pointwise stabiliser of $S \subseteq T$, or $\mathrm{Stab}_G(S)$ if there is no danger of confusing the action.

The *translation length* of an element $g \in G$ is defined as $|g| \coloneqq \inf_{x \in T} \mathrm{d}_T(x, gx)$. The *minset* of g is the subtree spanned by all points x which realise the translation length. If $|g| = 0$ the element is called *elliptic*, and its minset is the sub-tree of all fixed points of g. If otherwise $|g| > 0$ the element is *loxodromic*, and its minset is a geodesic line on which g acts by translations.

The following lemmas are straightforward:

Lemma 1.2 ([CV96, Lemma 6.11]). *Suppose a group G acts on a tree T, and let g and h be commuting elements. Then the minset of g is invariant under h.*

Lemma 1.3 ([Cla14, Corollary 2.5]). *If $\mathbb{Z}^2$ acts on a tree without a global fixed point, then for any basis $\{g, h\}$, one of the elements must act loxodromically.*

Recall also that a group G *splits* over a family $\mathcal{Z}$ of subgroups if G is isomorphic to the fundamental group of a *graph of groups* whose edge groups are conjugates of subgroups in $\mathcal{Z}$. If this is the case, then G acts on the *Bass-Serre tree* of the splitting, and edge stabilisers are conjugates of subgroups in $\mathcal{Z}$. If moreover G does not coincide with any vertex group (that is, if the splitting is non-trivial), then the action has no global fixed point.

Conversely, if a group G acts without global fixed points on a tree T, then G is isomorphic to the fundamental group of a (non-trivial) graph of groups, whose edge groups are conjugates of edge stabilisers. With a little abuse of notation, we often conflate a G-tree with the corresponding graph of groups decomposition.

2. Cyclic splittings of Artin groups

Given a finite, labelled simplicial graph Γ, let A_Γ be the associated Artin group. We fix once and for all an identification between Γ and a generating set of A_Γ, and we will always conflate a vertex of Γ with the corresponding group element.

We recall here the notion of a parabolic subgroup. A subgraph $\Lambda \leqslant \Gamma$ is said to be *full*, or *the full subgraph of Γ induced by* $\mathrm{V}(\Lambda)$, if every edge of Γ connecting two vertices of Λ is also an edge of Λ. For every full subgraph $\Lambda \leqslant \Gamma$ we can consider the subgroup $\langle \mathrm{V}(\Lambda) \rangle_{A_\Gamma}$ of A_Γ generated by $\mathrm{V}(\Lambda)$. Such a subgroup is called the *standard parabolic subgroup* of A_Γ generated by $\mathrm{V}(\Lambda)$ and is denoted $A_{\mathrm{V}(\Lambda)}$. Then $A_{\mathrm{V}(\Lambda)}$ is isomorphic to A_Λ, by a result of Van der Lek [VdL83, Theorem 4.13], so henceforth we shall denote the standard parabolic subgroup on Λ by A_Λ. A *parabolic subgroup* of A_Γ is a conjugate of a standard parabolic subgroup.

Recall that, if $\{a, b\}$ is an edge labelled with some integer $m \geqslant 3$, the *dihedral Artin group* $\mathrm{DA}_m \coloneqq \langle a, b \mid \mathrm{prod}(a, b, m) = \mathrm{prod}(b, a, m) \rangle$ has infinite cyclic centre, generated by

$$z_{ab} = \begin{cases} \Delta_{ab} & \text{if } m \text{ is even;} \\ \Delta_{ab}^2 & \text{if } m \text{ is odd,} \end{cases}$$

where $\Delta_{ab} = \mathrm{prod}(a, b, m)$ is the *Garside element* of the dihedral (see e.g. [BS72]).

We now move to the characterisation of when an Artin group splits over $\mathbb{Z}$. Throughout the paper we actually work in the general setting of splittings over virtually cyclic subgroups. In fact every splitting we encounter is over $\mathbb{Z}$, which is unsurprising since Artin groups are conjecturally torsion-free while a virtually cyclic group which is not $\mathbb{Z}$ must have finite-order elements (see [Mac96, Lemma 3.2]).

Recall that a vertex v of a simplicial graph Γ is *separating* if the subgraph spanned by $\mathrm{V}(\Gamma) - \{v\}$ is disconnected.

Theorem 2.1. *Let Γ be a finite, connected, labelled simplicial graph. The Artin group A_Γ splits over virtually cyclic groups if and only if*

- *Γ has two vertices, or*
- *Γ has a separating vertex.*

In both cases, A_Γ splits over $\mathbb{Z}$.

The "if" part of Theorem 2.1 is clear. Indeed, if Γ has at most two vertices, the corresponding Artin group is isomorphic to either $\mathbb{Z}^2$ or a dihedral Artin group, all of which split over $\mathbb{Z}$ (see Example 3.3 for explicit splittings of dihedral Artin groups over $\mathbb{Z}$). Moreover, if Γ has a separating vertex v, then there exist two proper induced subgraphs Γ_1 and Γ_2 of Γ such that $\Gamma_1 \cap \Gamma_2 = \{v\}$, while $\Gamma_1 \cup \Gamma_2 = \Gamma$; therefore A_Γ admits a *visual splitting* $A_{\Gamma_1} *_{\langle v \rangle} A_{\Gamma_2}$.

We shall devote the rest of the Section to the proof of the "only if" part, that is, if Γ has at least three vertices and no separating vertices, then A_Γ cannot split over virtually cyclic groups. It is useful to keep in mind that, by Bass-Serre theory, admitting no splitting over virtually cyclic groups is equivalent to the fact that, whenever the group acts on a tree without global fixed points, there exists an edge whose stabiliser is not virtually cyclic.

For the next lemma, recall that a *Hamiltonian cycle* in a graph is an embedded cycle that visits each vertex exactly once.

Lemma 2.2. *Let Γ be a finite simplicial graph admitting a Hamiltonian cycle. Then A_Γ does not split over virtually cyclic groups.*

Proof. Let T be a tree on which A_Γ acts without global fixed points, and we want to find an edge whose stabiliser is not virtually cyclic. Let C be the Hamiltonian

cycle, and let $\{v_1, \ldots, v_n, v_{n+1} = v_1\}$ be a cyclic ordering of its vertices. Whenever $m_j := m_{v_j v_{j+1}} \geqslant 3$ let $z_j := z_{v_j v_{j+1}}$. Let $g_1 = v_1$, and inductively define

$$g_{i+1} = \begin{cases} v_{j+1} & \text{if } g_i = v_j \text{ and } m_j = 2; \\ z_j & \text{if } g_i = v_j \text{ and } m_j \geqslant 3; \\ v_{j+1} & \text{if } g_i = z_j. \end{cases}$$

In other words, two consecutive g_i and g_{i+1} are either the vertices of an edge of C with label 2, or a vertex of an edge of C with label at least 3 and the generator of the corresponding centre. See Figure 2 for an example.
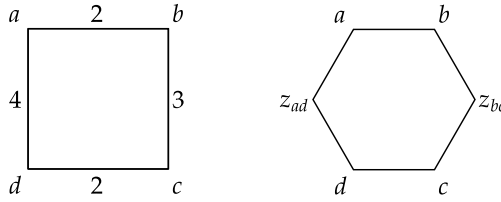

FIGURE 2. Here Γ is the square on the left. Then we set $g_1 = a$, $g_2 = b$, $g_3 = z_{bc}$, and so on. This way, every two consecutive g_i generate a copy of $\mathbb{Z}^2$.

Now define $G_i := \langle g_i, g_{i+1} \rangle \cong \mathbb{Z}^2$. By construction $G_{i-1} \cap G_i = \langle g_i \rangle$ (this is a consequence of [VdL83] if the subgroups are contained inside different dihedrals). There are two cases to analyse.

Case I. Assume first that each G_i fixes a point. If some G_i fixes an edge we are done; thus assume that each G_i fixes a unique point p_i. If all the p_i were the same, then A_Γ, which is generated by the G_is, would have a global fixed point. So let p be a leaf of the subtree S spanned by the p_i, and let $I \subseteq \{1, \ldots, r\}$ be such that G_i fixes p if and only if $i \in I$. Notice that I is non-empty, and does not contain all indexes as there is no global fixed point; hence there exist $j, k \in I$ such that $j - 1, k + 1 \notin P$, where addition is modulo r. Let p_{j-1} (resp. p_{k+1}) be the point fixed by G_{j-1} (resp. G_{k+1}). This way, $g_j \in G_{j-1} \cap G_j$ fixes the non-trivial segment $[p_{j-1}, p]$, and similarly $g_{k+1} \in G_k \cap G_{k+1}$ fixes $[p_{k+1}, p]$. As p was a leaf of S, the two segments share an edge, so $\langle g_j, g_{k+1} \rangle$ fixes an edge. Now, g_i and g_{k+1} are infinite order elements without a non-trivial common power, as they are either two different vertices, or a vertex and the generator of a centre, or the generators of the centres of two different dihedrals (in all three cases, the absence of a common power follows immediately from [VdL83]). In particular $\langle g_j, g_{k+1} \rangle$ is not virtually cyclic and fixes an edge, proving that A_Γ does not split over virtually cyclic groups.

Case II. Assume now that some G_i has no fixed points, say G_1 without loss of generality. By Lemma 1.3, we can assume without loss of generality that g_2 acts loxodromically, so it acts by translations on some geodesic line A. By Lemma 1.2, $g_1(A) = A$, so there exist integers k_1, k_2, with $k_1 \neq 0$, such that $h = g_1^{k_1} g_2^{k_2}$ fixes A pointwise. For the same reason, there exist integers l_2, l_3, with $l_3 \neq 0$, such that $h' = g_2^{l_2} g_3^{l_3}$ fixes A pointwise. Notice that, since $h \in G_1$, $h' \in G_2$, and $G_1 \cap G_2 = \langle g_2 \rangle$, we get that h and h' are infinite order elements without a non-trivial

common power. Hence, as in Case I, the fact that $\langle h, h' \rangle$ fixes an edge concludes the proof. $\qquad \square$

Lemma 2.3. *Let Γ be a finite, connected, labelled simplicial graph on at least three vertices, and let $\mathcal{L}$ be a collection of induced subgraphs of Γ such that:*

(1) *For every $\Lambda \in \mathcal{L}$, A_Λ does not split over virtually cyclic groups;*
(2) *Every two-edges segment of Γ belongs to some $\Lambda \in \mathcal{L}$.*

Then A_Γ does not split over virtually cyclic groups.

Proof. Suppose that A_Γ acts on a tree T, without a global fixed point. If some A_Λ has no fixed point, then by (1) a non-virtually-cyclic subgroup of A_Λ (and therefore of A_Γ) fixes an edge, and we are done. Thus assume that each A_Λ fixes a point. In particular, every vertex of Γ fixes a point (this is because Γ is connected and has at least three vertices, so every vertex belongs to a two-edges segment). Furthermore, every two-edges segment with vertices $\{a, b, c\}$ fixes a vertex, which we can assume to be unique as otherwise the non-virtually-cyclic subgroup $\langle a, b \rangle$ would fix an edge.

Since there is no global fixed point, there exist vertices v, v' of Γ which do not share a fixed point. As argued above, v and v' cannot belong to the same two-edges segment, so they are at distance $n \geqslant 3$ in Γ. Furthermore, if we take a combinatorial path $v = v_0, \ldots, v_n = v'$ from v to v', there must be some index $0 < i < n - 1$ such that $\langle v_{i-1}, v_i, v_{i+1} \rangle$ and $\langle v_i, v_{i+1}, v_{i+2} \rangle$ fix different points. In particular, the dihedral $\langle v_i, v_{i+1} \rangle$ fixes an edge, so A_Γ does not split over virtually cyclic groups. $\qquad \square$

The following concludes the proof of Theorem 2.1:

Lemma 2.4. *Let Γ be a finite, connected, labelled simplicial graph on at least three vertices, without separating vertices. Then A_Γ does not split over virtually cyclic groups.*

Proof. Let $\mathcal{L}$ be the collection of induced subgraphs of Γ spanned by embedded cycles. For every $\Lambda \in \mathcal{L}$, A_Λ does not split over virtually cyclic groups by Lemma 2.2; moreover, since Γ has at least three vertices and no separating vertex, every two-edges segment belongs to an embedded cycle. Hence $\mathcal{L}$ satisfies the requirements of Lemma 2.3, and the conclusion follows. $\qquad \square$

We end the Section with two remarks, respectively completing the picture of which Artin groups split over $\mathbb{Z}$, and reminding the reader of when an Artin group splits over finite subgroups.

Remark 2.5 (Disconnected graphs)**.** If Γ is disconnected then A_Γ splits over $\mathbb{Z}$. Indeed if Γ has at least three vertices then it admits a visual splitting over a vertex; if instead Γ consists of two vertices then $A_\Gamma \cong F_2$, which splits over $\mathbb{Z}$ as $\langle a, b, t \mid tat^{-1} = b \rangle \cong \langle a, t \rangle$.

Remark 2.6 (One-endedness)**.** Recall that, by Stallings' Theorem [Sta71], a group has more than one end if and only if it does not split over finite subgroups. For an Artin group A_Γ, the latter happens if and only if Γ is either a point or disconnected. Indeed, if Γ is disconnected then A_Γ is the free product of the parabolic subgroups on its connected components, while if Γ is a point then $\mathbb{Z}$ is a HNN extension of the trivial group. Conversely, [BDM09, Proposition 1.3] states that an Artin group on a connected graph on at least two vertices is not relatively hyperbolic, and in particular it cannot split over finite subgroups.

3. JSJ DECOMPOSITION OVER VIRTUALLY CYCLIC SUBGROUPS

This Section is devoted to the explicit construction of a JSJ decomposition over virtually cyclic subgroups for a one-ended Artin group. Recall that we work in the setting of Notation 1.1, so all group actions on trees are assumed to be minimal and without inversions.

Definition 3.1 (VC-splitting). Let G be a group. A VC-*splitting* of G is a G-tree (T, Ω) such that, for every edge e of T, $\mathrm{Stab}_\Omega(e)$ is virtually cyclic.

Definition 3.2 (JSJ tree, [GL17]). A VC-splitting (T, Ω) of a group G is a *JSJ tree over virtually cyclic subgroups* if:
- T is *universally elliptic*, meaning that, for every edge e of T and every VC-splitting (T', Ω'), $\mathrm{Stab}_\Omega(e)$ acts elliptically on T';
- T *dominates* every universally elliptic tree, meaning that, if T' is any universally elliptic VC-splitting, there is a G-equivariant map $T \to T'$; in other words, vertex stabilisers for Ω act elliptically on T'.

The associated graph of groups decomposition of G is called a *JSJ decomposition over virtually cyclic subgroups*.

Example 3.3 (JSJ splittings of dihedral Artin groups). We first exhibit a JSJ decomposition over virtually cyclic groups for a dihedral Artin group DA_n. Let (T, Ω) be a VC-tree for DA_n. Notice that edge stabilisers in Ω cannot be trivial, as dihedrals are one-ended by Remark 2.6; moreover, they must be infinite cyclic, since dihedrals are torsion-free (this follows from e.g. [CMV23, Theorem D]).

First, let n be odd. Consider the splitting (T_n, Ω_n) as an amalgamated product induced by the isomorphism $\mathrm{DA}_n \cong \langle x, y \mid x^2 = y^n \rangle$, where $y = ab$ and $x = ab\ldots a$. Let $z = x^2$, and suppose by contradiction that z acts loxodromically on T. Since z generates the centre, every element in DA_n commutes with z, so it must fix the axis of z by Lemma 1.2. In particular, by minimality of the action, the whole T coincides with the axis of z. Let g generate the stabiliser of an edge of T, and notice that, since the action is without inversions, g fixes the whole line pointwise. Then DA_n would be isomorphic to a semidirect product $\langle g \rangle \rtimes \langle z \rangle \cong \mathbb{Z}^2$, and this is not the case. We thus proved that z must act elliptically on T, and in turn so do x and y as they are roots of an elliptic element. This shows that (T_n, Ω_n) is a JSJ tree over cyclic groups, as its vertex stabilisers are generated by conjugates of x and y.

Now let $n = 2m$ for $m \geqslant 2$, and let (T_n, Ω_n) be the splitting as an HNN extension induced by the isomorphism $\mathrm{DA}_n \cong \langle x, y \mid xy^m x^{-1} = y^m \rangle$, where $y = ab$ and $x = a$. Let $z = y^m$, which generates the centre. By the above argument z must act elliptically on T, and in turn so does its root y, which generates the vertex group. Then again (T_n, Ω_n) is a JSJ tree over cyclic groups.

Finally, there is no JSJ decomposition over cyclic subgroups of $\mathbb{Z}^2$. Indeed, given any primitive element $a \in \mathbb{Z}^2$, complete it to a basis $\{a, b\}$ and consider the splitting as an HNN extension $\mathbb{Z}^2 \cong \langle a, b \mid bab^{-1} = a \rangle$ where the vertex group is $\langle a \rangle$. This means that, if there were a JSJ decomposition over cyclic subgroups, then its edge stabilisers would be contained in every cyclic subgroup, so they would be trivial; this would contradict that $\mathbb{Z}^2$ is one-ended.

Notation 3.4. For the rest of this Section let Γ be a finite, connected, labelled simplicial graph on at least three vertices, so that A_Γ does not split over finite subgroups by Remark 2.6.

Definition 3.5. A *big chunk* is a connected induced subgraph of Γ without separating vertices, which is maximal (with respect to inclusion) with these properties. A *big big chunk* is a big chunk on at least three vertices. A *(big) big chunk parabolic* is a subgroup of A_Γ conjugated to some A_Λ, where Λ is a (big) big chunk.

Notice that two big chunks can only overlap over a vertex, which must separate Γ; conversely, every separating vertex belongs to at least two big chunks.

Remark 3.6 (Big chunk parabolics are retracts). Let Λ be a big chunk of Γ. For every vertex $v \in \Gamma$ define $\rho(v) \in \Lambda$ to be the closest vertex in Γ to v; such vertex is unique, because Λ is a big chunk. Then mapping every generator v to $\rho(v)$ gives a homomorphism $\rho \colon A_\Gamma \to A_\Lambda$ which is the identity on A_Λ.

Definition 3.7 ($J(\Gamma)$). Consider the bipartite graph $\mathcal{B}(\Gamma)$ defined as follows:
- $\mathcal{B}(\Gamma)$ has one black vertex for every big chunk, and one white vertex for every separating vertex of Γ. We denote black vertices by the corresponding big chunks, and white vertices by the corresponding vertices of Γ.
- If v is a white vertex and Λ is a black vertex, there is an edge $\{v, \Lambda\}$ if and only if $v \in \Lambda$.

Next, recall that a *leaf* of Γ is an edge one of whose endpoints, called the *tip* of the leaf, has valence one. A leaf is *even* or *odd* according to the parity of its label; an even leaf is *toral* if the label is 2, and is *braided* otherwise. Now let $\mathcal{B}'(\Gamma)$ be the graph obtained from $\mathcal{B}(\Gamma)$ as follows:
- glue a one-edge loop to each black vertex corresponding to a toral leaf;
- for every braided even leaf Λ, add a red vertex which is only adjacent to Λ.

We are finally ready to describe the candidate JSJ decomposition $J(\Gamma)$. The base graph of $J(\Gamma)$ is $\mathcal{B}'(\Gamma)$. Vertex and edge groups are as follows:
- If v is a white vertex, the associated vertex group is $\langle v \rangle$.
- If Λ is a black vertex which is not an even leaf, the corresponding vertex group is A_Λ.
- Let $\langle a, b \rangle$ be a black vertex corresponding to a toral leaf with tip b. The associated vertex group is $\langle a \rangle$, and the stable letter of the loop is b, which commutes with a.
- Let $\langle a, b \rangle$ be a black vertex corresponding to an even leaf with label $2m \geqslant 4$ and tip b, and let $z = z_{ab}$. The black vertex group is $\langle a, z \rangle \cong \mathbb{Z}^2$; the red vertex group is generated by $r = ab$; and the edge between them identifies r^m with z. To understand the construction, one should recall that

$$\langle a, b \mid (ab)^m = (ba)^m \rangle \cong \langle a, z, r \mid [a, z] = 1, \, r^m = z \rangle \cong \mathbb{Z}^2 *_{\mathbb{Z}} \mathbb{Z}.$$

- Whenever $\{v, \Lambda\}$ is an edge of $\mathcal{B}(\Gamma)$, the associated edge group is $\langle v \rangle$, and inclusions are given by subgraph containments.

The whole construction is clarified with an example in Figure 3.

Remark 3.8 (Minimality of $J(\Gamma)$). By construction, whenever e is a leaf of $J(\Gamma)$ and w is its tip, the edge group H associated to e properly embeds in the vertex group V for w. This is because, if w is black then V is a big chunk parabolic, which is not cyclic; otherwise w is red, $V = \langle r \rangle$ for some element r, and the edge group is generated by a proper power of r. By [Bas93, Lemma 7.12], this condition, together with the fact that the decomposition is finite, implies that A_Γ acts minimally on the Bass-Serre tree of $J(\Gamma)$, so we are in the setting of Notation 1.1.
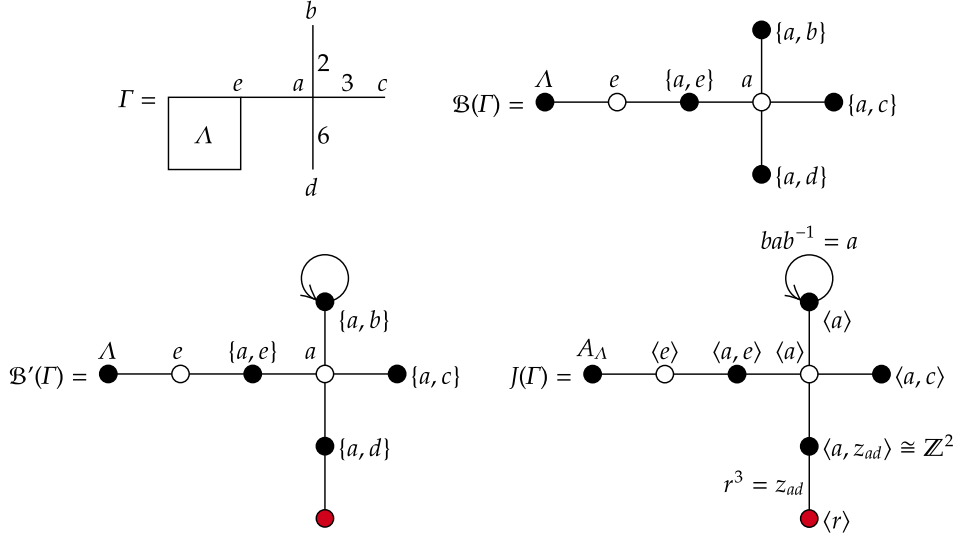
FIGURE 3. Example of the construction of the JSJ decomposition over virtually cyclic groups. First, one identifies the big chunks of Γ, which correspond to the black vertices of $\mathcal{B}(\Gamma)$. In this example, there are two even leaves, the toral leaf $\{a, b\}$ and the braided leaf $\{a, d\}$, with edge label 6. Therefore, in $\mathcal{B}'(\Gamma)$ we glue a loop to $\{a, b\}$ and a free edge to $\{a, d\}$. Next, every white vertex is labelled with the subgroup generated by the corresponding vertex of Γ. Every black vertex which is not an even leaf is labelled with the corresponding parabolic subgroup, while even leaves are further decomposed as HNN extensions (for toral leaves) or as amalgamated free products of $\mathbb{Z}^2$ and $\mathbb{Z}$ (for braided leaves). Every edge of $J(\Gamma)$ without a label represents the embedding of the white vertex group into the black vertex group, induced by inclusion of subgraphs of Γ.

Theorem 3.9. *Let Γ be a connected graph on at least three vertices. If T is a VC-tree, every vertex group of $J(\Gamma)$ acts elliptically on T. As a consequence, $J(\Gamma)$ is a JSJ-decomposition for A_Γ over virtually cyclic subgroups.*

Proof. We first record two facts which we shall use multiple times.

Claim 3.10. *Let $\{a, b, c\} \subseteq \Gamma$ be a two-edges segment, with $a \notin \mathrm{Link}_\Gamma(c)$. Then b acts elliptically on T.*

Proof of Claim 3.10. Let g be either a if $m_{ab} = 2$, or z_{ab} if $m_{ab} \geqslant 3$. Similarly, define h as c if $m_{bc} = 2$, or z_{bc} if $m_{bc} \geqslant 3$. If b were loxodromic then one could find $k_1, k_2, l_1, l_2 \in \mathbb{Z}$, with $k_1, l_1 \neq 0$, such that the non-virtually-cyclic subgroup $\langle g^{k_1} b^{k_2}, h^{l_1} b^{l_2} \rangle$ would fix an edge, as in the proof of Lemma 2.2. $\square$

With the same techniques, one gets:

Claim 3.11. *Let $\{a, b\}$ be an edge of Γ, with $m_{ab} \geqslant 3$. Then z_{ab} acts elliptically on T.*

Proof of Claim 3.11. Again, if z_{ab} acted loxodromically, one could find $k_1, k_2, l_1, l_2 \in \mathbb{Z}$, with $k_1, l_1 \neq 0$, such that the non-virtually-cyclic subgroup $\langle a^{k_1} z_{ab}^{k_2}, b^{l_1} z_{ab}^{l_2} \rangle$ would fix an edge. $\square$

Going back to the proof, we want to show that every vertex group in $J(\Gamma)$ acts elliptically on T. By construction, every white vertex group of $J(\Gamma)$ is a subgroup of a black vertex group; moreover, if $\langle a, b \rangle$ is a braided even leaf, the associated red vertex group is generated by a root r of z_{ab}, and in particular if z_{ab} acts elliptically on T then so does r. By this argument, we only need to consider black vertex groups. Furthermore, big big chunk parabolics must act elliptically, since by Lemma 2.4 they cannot split over virtually cyclic groups. So we are left to show that the vertex group associated to a big chunk on two vertices $\{a, b\}$ is elliptic. There are two cases to analyse.

CASE I: $\{a, b\}$ is not a leaf. In this setting both a and b are separating vertices, so there exist two edges $\{v, a\}$ and $\{b, w\}$ such that a separates v from b, and similarly b separates w from a. By Claim 3.10, a and b both act elliptically on T. Then the black vertex group associated to $\{a, b\}$, which is the dihedral $\langle a, b \rangle$, acts elliptically by the following Claim:

Claim 3.12. *Let $\{a, b\}$ be an edge of Γ. If a and b both act elliptically on T, then $\langle a, b \rangle$ fixes a point.*

Proof of Claim 3.12. If $m_{ab} = 2$ then $\langle a, b \rangle \cong \mathbb{Z}^2$ must act elliptically, as otherwise one of the generators should act loxodromically by Lemma 1.3. Then suppose that $m_{ab} \geqslant 3$, and let $z = z_{ab}$. By Claim 3.11 z acts elliptically, so let T' be the minset of z. Notice that both a and b act on T', by Lemma 1.2. Moreover, since both a and z act elliptically, the whole subgroup $\langle a, z \rangle$ fixes a point of T, by the contrapositive of Lemma 1.3; in other words, a (and symmetrically b) must fix a point in T'.

Now, for every $n \in \mathbb{Z} - \{0\}$, the fixed subtree $\mathrm{Fix}_{T'}(a^n)$ is a single point p_a not depending on n, because, if $\mathrm{Fix}_{T'}(a^n)$ properly contained the non-trivial subtree $\mathrm{Fix}_{T'}(a)$, then the non-virtually-cyclic subgroup $\langle a^n, z_{ab} \rangle$ would fix an edge. As a consequence, $\mathrm{Fix}_{T'}(a)$ coincides with the *stable* fixed point set

$$\{p_a\} = \mathrm{Fix}_{T'}^\infty(a) \coloneqq \bigcup_{n \in \mathbb{N} - \{0\}} \mathrm{Fix}_{T'}(a^n).$$

A similar argument shows that $\mathrm{Fix}_{T'}^\infty(b) = \mathrm{Fix}_{T'}(b)$ is a single point p_b. Now suppose that a and b do not fix a common vertex, and we claim that this leads to the contradiction that a and b generate a free group. Indeed, let $\gamma = [p_a, p_b] \subset T'$, and decompose $T' = X_a \cup \gamma \cup X_b$ where $X_a \cap \gamma = \{p_a\}$ and similarly $X_b \cap \gamma = \{p_b\}$. Since $\{p_a\} = \mathrm{Fix}_{T'}^\infty(a)$, we see that $a^n(X_b) \subset X_a$ for every $n \in \mathbb{Z} - \{0\}$, and symmetrically with a and b swapped. Then the ping-pong lemma (see e.g. [Man17]) shows that $\langle a, b \rangle$ is non-abelian free, a contradiction. $\square$

CASE II: $\{a, b\}$ is a leaf. Let b be the tip of the leaf. Since Γ has at least three vertices and is connected, there exists $c \in \mathrm{Link}_\Gamma(a) - \{b\}$. Applying Claim 3.10 to the two-edges segment $\{b, a, c\}$ shows that a acts elliptically on T. There are now three cases to analyse, depending on m_{ab}.

- If $m_{ab} = 2$, the black vertex group associated to the leaf is $\langle a \rangle$, and there is nothing else to prove.

- If $m_{ab} > 2$ is even then z_{ab} acts elliptically by Claim 3.11; thus the black vertex group associated to the leaf, which is $\langle a, z_{ab} \rangle \cong \mathbb{Z}^2$, acts elliptically by Lemma 1.3.
- If $m_{ab} > 2$ is odd then b acts elliptically as well, since it is conjugated to a; hence the black vertex group associated to the leaf, which is $\langle a, b \rangle$, fixes a point by Claim 3.12.

The proof of Theorem 3.9 is now complete. $\qquad\square$

4. Acylindrical hyperbolicity of automorphism groups

In this Section we prove that the automorphism group of an Artin group A_Γ is acylindrically hyperbolic, provided the existence of a separating vertex of Γ which is not central in A_Γ. To do so, from the JSJ decomposition from Section 3 we shall build an $\mathrm{Aut}(A_\Gamma)$-invariant tree, which is an instance of the so-called tree of cylinders from [GL11].

4.1. **Background on acylindrical hyperbolicity.** The notion of acylindricity is due to Sela, who originally formulated it for actions on trees, and was then extended by Bowditch to action on general metric spaces [Sel97, Bow08].

Definition 4.1 (Acylindricity). Let G be a group and let (X, d) be a metric space. An action of G on (X, d) by isometries is *acylindrical* if, for every $\varepsilon \in \mathbb{R}_{\geqslant 0}$ there exist $L \in \mathbb{R}_{\geqslant 0}$ and $N \in \mathbb{N}$ such that, for every $x, y \in X$, if $\mathrm{d}(x, y) \geqslant L$, then

$$|\{g \in G : \mathrm{d}(x, gx) < \varepsilon \text{ and } \mathrm{d}(y, gy) < \varepsilon\}| \leqslant N.$$

In the case where (X, d) is a simplicial tree endowed with the standard metric (i.e. edges are assigned unit length), this definition is equivalent to Sela's original one [Bow08, Section 2]: there exist $L, N \in \mathbb{N}$ such that, for every $u, v \in \mathrm{V}(X)$, if $\mathrm{d}(u, v) \geqslant L$, then $\mathrm{Stab}_G(u) \cap \mathrm{Stab}_G(v)$ has order bounded by N. In other words, an action on a tree X is acylindrical if geodesic segments of length at least L are fixed by at most N elements.

Recall that a geodesic metric space X is *hyperbolic* if there exists $\delta \geqslant 0$ such that, for every geodesic triangle in X, each side is contained in the δ-neighbourhood of the union of the other two.

Definition 4.2 ([Osi16]). A group is *acylindrically hyperbolic* if it is not virtually cyclic and it admits an acylindrical action on a hyperbolic metric space with unbounded orbits.

4.2. **Trees of cylinders.** We now recall the construction of the tree of cylinders from [GL11]. For experts, we shall specialise the general definition to the case of virtually cyclic subgroups, with commensurability as the so-called admissible relation.

Definition 4.3 (Cylinder). Let (T, Ω) be a VC-tree for a group G. Given an edge e of T, its *cylinder* C_e is the subforest of all edges e' such that $\mathrm{Stab}_\Omega(e)$ and $\mathrm{Stab}_\Omega(e')$ are commensurable (i.e. their intersection has finite-index in both).

By [GL11, Lemma 4.2], a cylinder is actually a subtree, so two cylinders can overlap on at most a vertex.

Definition 4.4 (Tree of cylinders). The *tree of cylinders* of (T, Ω) is the bipartite tree T_c with vertex set $\mathrm{V}(T_c) = V_0 \sqcup V_1$ defined as follows:

- V_0 is the set of vertices x of T belonging to at least two distinct cylinders;
- V_1 is the set of cylinders of T;
- there is an edge $e = \{x, C\}$ between $x \in V_0$ and $C \in V_1$ if and only if x (viewed as a vertex of T) belongs to C (viewed as a subtree of T).

As G acts on the set of cylinders, there is a natural isometric G-action on T_c which we denote by Ω_c. It is clear from the construction that $\mathrm{Stab}_{\Omega_c}(x) = \mathrm{Stab}_\Omega(x)$ for every $x \in V_0$; moreover, for every edge e of T we have that $\mathrm{Stab}_{\Omega_c}(C_e)$ is the *commensurator* of $\mathrm{Stab}_\Omega(e)$ in G, i.e. the set of all $g \in G$ such that $\mathrm{Stab}_\Omega(e)$ and $g\,\mathrm{Stab}_\Omega(e)\,g^{-1}$ are commensurable.

It is easy to see that T_c is indeed a tree [Gui04]; moreover, by [Gui04, Lemma 4.9], Ω_c is minimal if Ω is minimal.

The following proposition is known to experts and included for completeness. The proof is largely based on personal communications with Yassine Guerch and Gilbert Levitt, which we both thank.

Proposition 4.5 (see [GL11])**.** *Let G be a group whose torsion subgroups have uniformly bounded cardinality. Let (T, Ω) be a JSJ-tree over virtually cyclic subgroups of G which is not a point. Then G acts minimally and acylindrically on T_c, and the action extends to an isometric action of $\mathrm{Aut}(G)$.*

Proof. We already noticed that the G-action is minimal, and it extends to $\mathrm{Aut}(G)$ by [GL11, Corollary 4.10], so we are left to prove acylindricity. Let $p, q \in V(T_c)$ be at distance at least 6, and we claim that $\mathrm{Stab}_{\Omega_c}(p) \cap \mathrm{Stab}_{\Omega_c}(q)$ has uniformly bounded cardinality. Since the tree is bipartite, we can find $p', q' \in V_0$ which lie on a geodesic $[p, q]$ and are at distance at least 4. Since any element fixing p and q must also fix p and q', it is enough to bound the cardinality of $\mathrm{Stab}_{\Omega_c}(p') \cap \mathrm{Stab}_{\Omega_c}(q')$. Since $p', q' \in V_0$, the latter equals $H \coloneqq \mathrm{Stab}_\Omega(p') \cap \mathrm{Stab}_\Omega(q')$, which is therefore virtually cyclic as it is contained inside the stabiliser of some edge of T. If H is finite then we are done, as torsion subgroups of G have uniformly bounded cardinality. Otherwise H is commensurable to all edge stabilisers on a T-geodesic between p' and q', which therefore belong to the same cylinder, violating the fact that $\mathrm{d}_{T_c}(p', q') \geqslant 4$. $\square$

4.3. **Application to automorphism groups of Artin groups.**

Corollary 4.6. *Let A_Γ be a torsion-free Artin group such that Γ is connected and has a separating vertex s which does not centralise the whole group. Then $\mathrm{Aut}(A_\Gamma)$ is acylindrically hyperbolic.*

Proof. By hypothesis, there is a vertex $t \in \Gamma$ which generates either a non-abelian dihedral or a free group with s. It is easy to see that t does not commensurate $\langle s \rangle$, so $\langle s \rangle$ is *weakly malnormal*, meaning that it has finite (in this case, trivial) intersection with one of its conjugates. Then, since A_Γ splits as an amalgamated product over $\langle s \rangle$, it is acylindrically hyperbolic by [MO15, Corollary 2.2]. In turn, since it is also torsion-free, [Bog22, Corollary E] provides an element w such that, if an endomorphism $\varphi \colon A_\Gamma \to A_\Gamma$ fixes w, then φ is the conjugation by a power of w.

Now let (T, Ω) be the Bass-Serre tree of the JSJ splitting $J(\Gamma)$ from Definition 3.7, which is not a point since Γ is not a single big chunk. By Proposition 4.5, A_Γ has a minimal, acylindrical action on the tree of cylinders T_c which extends to $\mathrm{Aut}(A_\Gamma)$. We take a small detour to prove the following:

Claim 4.7. *The element w acts loxodromically on T_c, which is therefore unbounded.*

Proof of Claim 4.7. Towards a contradiction, suppose that w fixes some vertex of T_c. If it fixes some $x \in V_0$ then $w \in \mathrm{Stab}_\Omega(x)$, which is a vertex group of T and is therefore contained in a big chunk parabolic H. However, by Remark 3.6 there is a retraction $\rho \colon A_\Gamma \to H$, which is absurd as ρ fixes w and is not a conjugation.

Now assume that w fixes the cylinder C_e, for some edge e of T. We want to show that w acts elliptically on T, from which we shall deduce a contradiction with the exact same argument as above.

Suppose first that e connects a black and a red vertex, so that $\mathrm{Stab}_\Omega(e) = g\langle z_{ab}\rangle g^{-1}$ for some braided even leaf $\{a,b\}$ and some $g \in A_\Gamma$. Up to replacing w by gwg^{-1}, which has the same characterisation as w in terms of endomorphisms, we can assume that $g = 1$. Since w commensurates $\mathrm{Stab}_\Omega(e)$, there are non-trivial integers $m, n \in \mathbb{Z} - \{0\}$ such that $wz_{ab}^m w^{-1} = z_{ab}^n$. By looking at the retraction $\rho \colon A_\Gamma \to \langle a, b\rangle$, where z_{ab} is central, we see that $m = n$, so w and z_{ab}^n commute. But then the conjugation ψ by z_{ab}^n fixes w, so ψ must coincide with the conjugation by a power of w. Hence it must be that w is a root of z_{ab}^n, since a torsion-free acylindrically hyperbolic group is centerless [Osi16, Corollary 7.2]. This means that w acts elliptically on T as one of its powers does, as required.

We are left to consider the case where e is an edge whose stabiliser is conjugate to $\langle s\rangle$, for some separating vertex s. As above, up to conjugating w we can assume that w commensurates $\langle s\rangle$, so there are non-trivial integers $m, n \in \mathbb{Z} - \{0\}$ such that $ws^m w^{-1} = s^n$. By looking at the map $\rho \colon A_\Gamma \to \mathbb{Z}$ sending every generator to 1 we see that $m = n$, so exactly as above we get that w is a root of s^n, and in particular acts elliptically on T, as required. $\qquad\square$

Finally, $\mathrm{Aut}(A_\Gamma)$ is acylindrically hyperbolic since the $\mathrm{Aut}(A_\Gamma)$-action on T_c satisfies the requirements of [GH21, Proposition 3.8]:

- $\mathrm{Aut}(A_\Gamma)$ is not virtually cyclic as it contains A_Γ (here we are again using that A_Γ has no centre);
- The A_Γ-action on T_c is acylindrical and minimal, hence orbits are unbounded as so is T_c. Thus the action is non-elementary by [Osi16, Theorem 1.1];
- w is WPD, as it is a loxodromic element in an acylindrical action. $\qquad\square$

5. Isomorphism invariance of big chunks

Using the JSJ decompositions we constructed, in this final Section we prove that, if two Artin groups are isomorphic, then any isomorphism must preserve the isomorphism type of big chunks. In fact, we show that every isomorphism preserves the conjugacy class of any big chunk that is not a toral leaf.

We start with a lemma about dihedral Artin groups:

Lemma 5.1. *Let $\langle a, b \mid (ab)^m = (ba)^m\rangle$ be a dihedral Artin group with label $2m$. If $x \in \langle a, z_{ab}\rangle$ is a primitive element admitting an n-th root $w \in \langle a, b\rangle$, then $n \leqslant m$.*

Proof. Consider the presentation of the dihedral as the Baumslag-Solitar group $\langle a, r \mid [a, r^m] = 1\rangle \cong \mathbb{Z}*_{m\mathbb{Z}}$, where $r = ab$. In the Bass-Serre tree of this HNN extension, $\langle a, z_{ab}\rangle = \langle a, r^m\rangle$ is the setwise stabiliser of the axis A of a. If w acts loxodromically then its axis, which is also the axis of $x = w^n \in \langle a, r^m\rangle$, must be fixed setwise by a, as a consequence of Lemma 1.2, so it must coincide with A. In this case w fixes A setwise, so $w \in \langle a, r^m\rangle$ against the assumption that w was a non-trivial root. If instead w is elliptic then so is x, and the only elliptic elements in

$\langle a, r^m \rangle$ belong to $\langle r^m \rangle$ (this follows from Lemma 1.3). Since x is primitive, we must have that $x = r^{\pm m}$, so w has order n in the central quotient $\langle a, b \rangle / \langle r^m \rangle \cong \mathbb{Z} * \mathbb{Z}/m\mathbb{Z}$. It now suffices to notice that torsion elements in the latter have order at most m. $\square$

From now on, all defining graphs are finite, connected and on at least three vertices, as in Notation 3.4. We now introduce another cyclic splitting of an Artin group, which corresponds to the maximal visual splitting over separating vertices:

Definition 5.2. Let $\overline{J(\Gamma)}$ be the splitting obtained from $J(\Gamma)$ by collapsing to a point every edge that either has a red vertex as an endpoint, or is a loop. See Figure 4 for an example of how to obtain $\overline{J(\Gamma)}$ from $J(\Gamma)$.
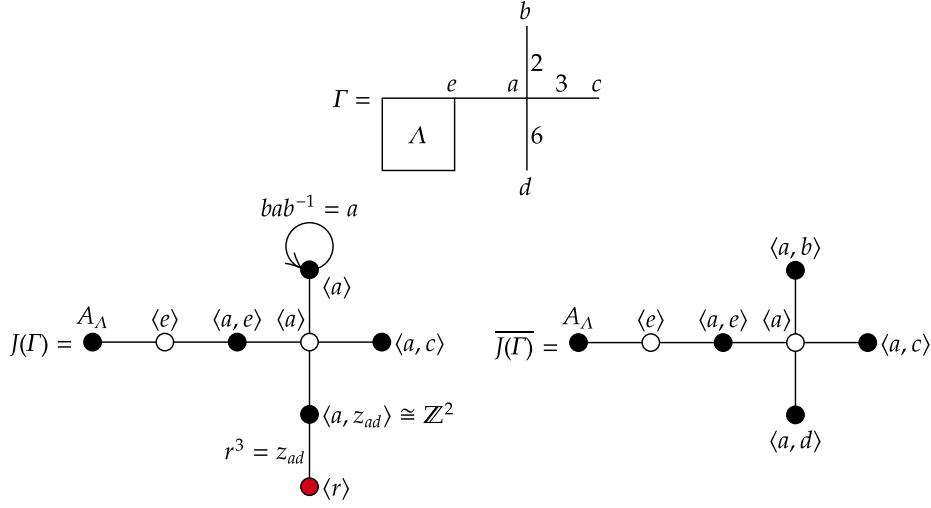

FIGURE 4. $\overline{J(\Gamma)}$ has one white vertex for every separating vertex of Γ, and one black vertex for every big chunk. Now all vertices of Γ act elliptically, including the tips of even leaves.

Lemma 5.3. *Let H, V be two standard big chunk parabolics of A_Γ. If $H \leqslant kVk^{-1}$ for some $k \in A_\Gamma$, then $H = V$ and $k \in H$.*

Proof. In the Bass-Serre tree of $\overline{J(\Gamma)}$, the non-cyclic subgroup H has a unique fixed point, which is the coset H; similarly, kVk^{-1} only fixes kV. However kVk^{-1} contains H, so $kVk^{-1} \leqslant \text{Stab}_{A_\Gamma}(H) = H$. Hence $H = kVk^{-1}$, and therefore their fixed cosets H and kV coincide. In turn this means that $k \in V$, so $H = V$. $\square$

The exact same proof, with $J(\Gamma)$ replacing $\overline{J(\Gamma)}$, yields the following:

Lemma 5.4. *Let H, V be two non-cyclic vertex groups of $J(\Gamma)$. If $H \leqslant kVk^{-1}$ for some $k \in A_\Gamma$, then $H = V$ and $k \in H$.*

We next notice that, since two JSJ decomposition have the same elliptic subgroups, then any isomorphism must preserve non-cyclic vertex groups:

Proposition 5.5. *Let $\varphi \colon A_\Gamma \to A_{\Gamma'}$ be an isomorphism. Suppose that $H \leqslant A_\Gamma$ is a non-cyclic black vertex group in $J(\Gamma)$. Then $\varphi(H)$ is conjugate to a black vertex group of $J(\Gamma')$.*

Proof. By Theorem 3.9, $\varphi(H)$ must act elliptically on $J(\Gamma')$; thus there exists a vertex group V of $J(\Gamma')$ and some $h \in A_{\Gamma'}$ such that $\varphi(H) \leqslant hVh^{-1}$. Since H is non-cyclic by assumption, V itself must be a non-cyclic black vertex group of $J(\Gamma')$. Repeating this argument with V and the inverse isomorphism φ^{-1}, we get

$$(1) \qquad H = \varphi^{-1}(\varphi(H)) \leqslant \varphi^{-1}(h)\varphi^{-1}(V)\varphi^{-1}(h)^{-1} \leqslant k\widetilde{H}k^{-1},$$

for some $k \in A_\Gamma$ and some non-cyclic black vertex group $\widetilde{H}$ of $J(\Gamma)$. By Lemma 5.4 all inequalities in Equation (1) are indeed equalities, so we must have had that $\varphi(H) = hVh^{-1}$. □

For the next Theorem, given a finite labelled graph Γ, let $\mathcal{BC}(\Gamma)$ be the set of big chunks of Γ.

Theorem 5.6 (Big chunks are isomorphism invariants). *Let Γ and Γ' be finite, connected, labelled simplicial graphs, and let $\varphi\colon A_\Gamma \to A_{\Gamma'}$ be an isomorphism. Then there exists a bijection $\varphi_\#\colon \mathcal{BC}(\Gamma) \to \mathcal{BC}(\Gamma')$ such that, for every $\Lambda \in \mathcal{BC}(\Gamma)$:*

(1) $A_\Lambda \cong A_{\varphi_\#(\Lambda)}$.
(2) If Λ is not a toral leaf, then $A_{\varphi_\#(\Lambda)}$ is a conjugate of $\varphi(A_\Lambda)$.
(3) If Λ is an even leaf, then so is $\varphi_\#(\Lambda)$.

Moreover, if φ maps standard generators of Γ to conjugates of standard generators of Γ', then we can arrange that $A_{\varphi_\#(\Lambda)}$ is a conjugate of $\varphi(A_\Lambda)$ for every $\Lambda \in \mathcal{BC}(\Gamma)$.

Proof. By e.g. [Vas22, Corollary B], if Γ has at most two vertices then Γ and Γ' are isomorphic as labelled graphs, so the result is trivial. Then assume that both Γ and Γ' have at least three vertices, so that we are in the setting of Notation 3.4 and we can use the JSJ decompositions from Definition 3.7.

We shall progressively construct $\varphi_\#$ on increasing subsets of $\mathcal{BC}(\Gamma)$, while checking at every step that conditions (2) and (3) are satisfied, and that the map is injective. Then the final map $\varphi_\#$ will be injective as well, and satisfy the conditions. As a consequence, $\varphi_\#$ will also be bijective, as the same procedure will also produce an injection $(\varphi^{-1})_\#$ in the opposite direction.

Let $\Lambda \in \mathcal{BC}(\Gamma)$, and let H be the corresponding black vertex group in $J(\Gamma)$. Suppose first that Λ is neither an even leaf nor an edge with label 2, and let $\mathcal{BC}^+(\Gamma)$ be the collection of such big chunks. By construction, $H = A_\Lambda$, which is not isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}^2$. By Proposition 5.5, $\varphi(H)$ is conjugate to a black vertex group of $J(\Gamma')$, which must be of the form $A_{\Lambda'}$ for some $\Lambda' \in \mathcal{BC}^+(\Gamma')$ as $\varphi(H)$ is also not isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}^2$. Thus we can define $\varphi_\#$ on $\mathcal{BC}^+(\Gamma)$ by setting $\varphi_\#(\Lambda) = \Lambda'$. This map is injective because if $\Lambda, \Delta \in \mathcal{BC}^+(\Gamma)$ then A_Λ and A_Δ are not conjugate by Lemma 5.3, so they map to non-conjugate vertex groups.

Assume next that $\Lambda = \{a, b\}$ is an edge of label 2 which is not a leaf, so that $H = A_\Lambda \cong \mathbb{Z}^2$. Again, Proposition 5.5 provides some black vertex group $V \cong \mathbb{Z}^2$ of $J(\Gamma')$ such that $\varphi(H)$ is conjugate to V. Suppose by contradiction that $V = \langle c, z_{cd} \rangle$ for some braided leaf $\{c, d\}$ with label $2m$. Then $z_{cd} = (cd)^m$, and since the isomorphism φ maps H to V there would be a primitive element $x \in H$ and some element $r \in A_\Gamma$ such that $r^m = x$. Now let $\rho\colon A_\Gamma \to H$ be the retraction from Remark 3.6, and notice that the element $\rho(r) \in H$ satisfies $\rho(r)^m = \rho(x) = x$, against the assumption that x was primitive in H.

By exclusion, we must have that $V = A_{\Lambda'}$ for some edge of label 2 which is not a leaf, so we can extend $\varphi_\#$ by setting $\varphi_\#(\Lambda) = \Lambda'$. Again by Lemma 5.3,

two different edges with label 2 which are not leaves must support non-conjugate parabolics, so the extension of $\varphi_\#$ is again injective.

Now let $\Lambda = \{a, b\}$ be a braided even leaf with label $2m$ and tip b, so that $H = \langle a, z_{ab}\rangle$. By the above argument applied to the inverse of φ, we see that $\varphi(H) = kVk^{-1}$, where $k \in A$ and $V = \langle c, z_{cd}\rangle$ for some braided even leaf $\Lambda' = \{c, d\}$ with label $2n$ and tip d.

We first show that $m = n$. Suppose by contradiction that this is not the case, and up to replacing φ by its inverse we can assume that $m < n$. Since φ conjugates H to V, to get a contradiction it is enough to show that a primitive element $x \in \langle a, z_{ab}\rangle$ cannot admit an n-th root $y \in A_\Gamma$. Let $\pi\colon A_\Gamma \to \langle a, b\rangle$ be the retraction from Remark 3.6. Then $\pi(y) \in \langle a, b\rangle$ is an n-th root of x, which by Lemma 5.1 means that $n \leqslant m$, a contradiction.

We now claim that $\varphi(\langle a, b\rangle) \leqslant k\langle c, d\rangle k^{-1}$. If this is true, then the same argument applied to φ^{-1} will give that $\varphi^{-1}(\langle c, d\rangle)$ is conjugated inside $\langle a, b\rangle$, so

$$\langle a, b\rangle = \varphi^{-1}(\varphi(\langle a, b\rangle)) \leqslant \varphi^{-1}(k\langle c, d\rangle k) \leqslant h\langle a, b\rangle h^{-1},$$

for some $h \in A$. Lemma 5.3 will then imply that the above containments are indeed equalities, so that $\varphi(\langle a, b\rangle) = k\langle c, d\rangle k^{-1}$. We will therefore set $\varphi_\#(\Lambda) = \Lambda'$, and again invoke Lemma 5.3 to get that the extension of $\varphi_\#$ is injective.

To see that $\varphi(\langle a, b\rangle) \leqslant k\langle c, d\rangle k^{-1}$, let ψ be the composition of φ and the conjugation by k^{-1}, so that we have to show that $\psi(\langle a, b\rangle) \leqslant \langle c, d\rangle$. In turn, it is enough to prove that $\psi(r) \leqslant \langle c, d\rangle$ where $r = ab$. Since $r^m = z_{ab}$, the element $x \coloneqq \psi(r)^m$ is primitive inside $\langle c, z_{cd}\rangle$; in particular, $\psi(r)$ is elliptic and fixes some coset gW, where $g \in A_{\Gamma'}$ and W is a vertex group of $J(\Gamma')$. Let $\mathcal{T}$ be the sub-tree of $J(\Gamma')$ corresponding to the $\langle c, d\rangle$-orbits of the edge between the black vertex $\langle c, z_{cd}\rangle$ and the red vertex $\langle cd\rangle$. If $gW \in \mathcal{T}$ we are done, because then both g and W belong to $\langle c, d\rangle$ and so $\psi(r) \in gWg^{-1} \leqslant \langle c, d\rangle$. Otherwise, x fixes the geodesic from gW to $\langle c, z_{cd}\rangle$, and on this geodesic let e be the last edge not contained in $\mathcal{T}$. By how $J(\Gamma')$ is constructed, we must have that $e = h\langle c\rangle$, where h lies in some $\langle c, d\rangle$-translate of $\langle c, z_{cd}\rangle$, and in particular in $\langle c, d\rangle$. Then $x \in h\langle c\rangle h^{-1} \cap \langle c, z_{cd}\rangle$. Since no power of hch^{-1} is central in $\langle c, d\rangle$, the image $\overline{x}$ of x in the central quotient $\langle c, d\rangle/\langle z_{cd}\rangle \cong \mathbb{Z} * \mathbb{Z}/m\mathbb{Z}$ is non-trivial, and lies in the intersection between $\langle \overline{c}\rangle$ and $\langle \overline{h}\overline{c}\overline{h}^{-1}\rangle$. Since $\langle \overline{c}\rangle$ generates the $\mathbb{Z}$-factor in the free product, which is malnormal, we have that $\overline{h} \in \langle \overline{c}\rangle$, so $h \in \langle c, z_{cd}\rangle$ and therefore $hch^{-1} = c$. In turn, $x \in \langle c\rangle$ is primitive, so $x = c^{\pm 1}$; however, sending every standard generator to 1 defines a map $A_{\Gamma'} \to \mathbb{Z}$, which maps $\psi(r)$ to an m-th root of ± 1, a contradiction.

We are left to define $\varphi_\#$ on toral leaves of Γ, whose number corresponds to the rank of the fundamental group of the graph underlying $J(\Gamma)$ (this rank is known as the *Betti number* of $J(\Gamma)$). By a combination of [For01, Theorem 1.1] and [GL17, Section 4], the two decompositions $J(\Gamma)$ and $J(\Gamma')$ have the same Betti number (in fact, they belong to the same *deformation space*). Thus we can fix any bijection ψ between the set of toral leaves of Γ and Γ', and for every such leaf Λ we set $\varphi_\#(\Lambda) = \psi(\Lambda)$. This completes the construction of $\varphi_\#$, which is injective as it was injective at every step.

In the setting of the "moreover" part of the statement we can modify the definition of $\varphi_\#$ on toral leaves. Let $\{a, b\}$ be a toral leaf with tip b. Since $J(\Gamma)$ and $J(\Gamma')$ have the same elliptic subgroups, $\varphi(b)$ must act loxodromically on $J(\Gamma')$,

so it must be conjugate to the tip d of some even leaf $\{c, d\}$ because all other vertices of Γ' act elliptically. Furthermore, such leaf must be toral, because φ preserves even braided leaves as we argued above. Hence let $k \in A$ be such that $\varphi(b) = kdk^{-1}$. By looking at the action of A_Γ on the Bass-Serre tree of the visual splitting $A_\Gamma \cong A_{\Gamma-\{a\}} *_{A_a} A_{\{a,b\}}$ one can see that $\mathrm{C}_{A_\Gamma}(b) = \langle a, b \rangle$: if $g \in A_\Gamma$ centralises b, then it stabilises the minset of b, which consists of the single vertex $\langle a, b \rangle$. Analogously, $\mathrm{C}_{A_{\Gamma'}}(d) = \langle c, d \rangle$. It then follows that

$$\varphi(\langle a, b \rangle) = \varphi(\mathrm{C}_{A_\Gamma}(b)) = \mathrm{C}_{A_{\Gamma'}}(\varphi(b)) = k\,\mathrm{C}_{A_{\Gamma'}}(d)k^{-1} = k\langle c, d \rangle k^{-1};$$

thus we can define $\varphi_\#(\{a, b\}) = \{c, d\}$, and again injectivity follows from Lemma 5.3.
$\square$

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